(2-1-1) Limits:

DEFINITION

Informal Definition of Limit

If the values of a function f of x approach the value L as x approaches c, we say f has **limit** L as x approaches c and we write

$\lim_{x \to c} f(x) = L$ (Read "The limit of f of x as x approaches c equals L")

EXAMPLE 1 As Table 2.1 and Fig. 2.1 suggest,

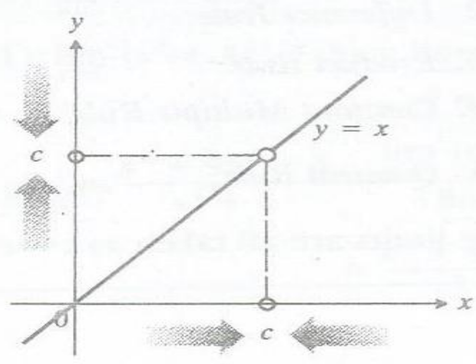
$$\lim_{x \to 2} (2x + 1) = 5$$

TABLE 2.1
Values of $f(x) = 2x + 1$ as $x \to 2$

Inputs approach 2 from the left							Inputs approach 2 from the right			
1.9	1.99	1.999	1.9999	⋯	2	⋯	2.0001	2.001	2.01	2.1
4.8	4.98	4.998	4.9998	⋯	5	⋯	5.0002	5.002	5.02	5.2
Outputs approach 5							Outputs approach 5			

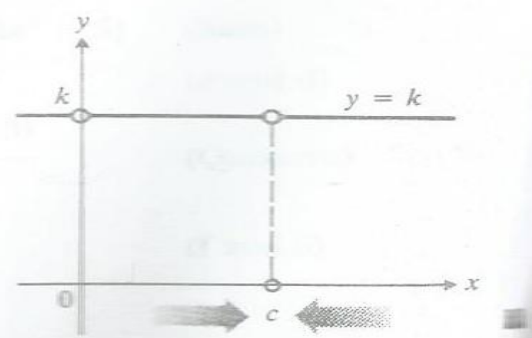
EXAMPLE 2 If f is the **identity function** $f(x) = x$, then for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x) = c$$



EXAMPLE 3 If f is the **constant function** $f(x) = k$ (the function whose outputs have the constant value k), then for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} (k) = k.$$



(2-1-2) Theorem of Limits:

Properties of Limits

THEOREM 1

Properties of Limits

If $\lim_{x\to c} f_1(x) = L_1$ and $\lim_{x\to c} f_2(x) = L_2$, then

1. *Sum Rule:* $\lim [f_1(x) + f_2(x)] = L_1 + L_2$
2. *Difference Rule:* $\lim [f_1(x) - f_2(x)] = L_1 - L_2$
3. *Product Rule:* $\lim f_1(x) \cdot f_2(x) = L_1 \cdot L_2$
4. *Constant Multiple Rule:* $\lim k \cdot f_2(x) = k \cdot L_2$ (any number k)
5. *Quotient Rule:* $\lim \dfrac{f_1(x)}{f_2(x)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$.

The limits are all taken as $x \to c$, and L_1 and L_2 are real numbers.

6- $\lim\limits_{x\to c} \sqrt[n]{f_{1(x)}} = \sqrt[n]{\lim\limits_{x\to c} f_{1(x)}} = \sqrt[n]{L_1}$.

7- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

8- $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$

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THEOREM 2:

Limit of polynomials can be found by substitution:

If f(x)=$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

Is any polynomial function then:

$$\lim_{x\to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0$$

THEOREM 3:

Limits of (many of not all)Rational Function can be found by substitution .

$\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{f(c)}{g(c)} = \quad (provided\ g(c) \neq 0).$

Ex4:

a) $\lim\limits_{x\to 3} x^2(2-x) = \lim\limits_{x\to 3}(2x^2 - x^3) = 2(3)^2 - (3)^3 = 18 - 27 = -9.$

b) Same limit, found another way:

$$\lim_{x\to 3} x^2(2-x) = \lim_{x\to 3} x^2 \cdot \lim_{x\to 3}(2-x) = (3)^2 \cdot (2-3) = 9\cdot(-1) = -9$$

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Ex5:

$$\lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{(2)^2+2(2)+4}{2+2} = \frac{12}{4} = 3.$$

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Ex6:

Find $\lim\limits_{x\to 2}\frac{x^3-8}{x^2-4} = \lim\limits_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim\limits_{x\to 2}\frac{(x^2+2x+4)}{(x+2)} = \frac{12}{4} = 3$

Ex7:

Find $\lim\limits_{x\to 5}\frac{x^2-25}{3(x-5)} = \lim\limits_{x\to 5}\frac{(x-5)(x+5)}{3(x-5)} = \lim\limits_{x\to 5}\frac{(x+5)}{3} = \frac{10}{3}$

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Ex8:

If f(x)=$\begin{cases} x, & x \neq 2 \\ 3, & x = 2 \end{cases}$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x) = 2$$

While f(2)=3

(2-1-3) Right-hand Limits and Left-hand limits:

Relationship between One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c^-} f(x) = L. \qquad (3)$$

EXAMPLE 9: **Show that** $\lim_{x\to 0} |x| = 0$

Solution:

We prove that $\lim_{x\to 0} |x| = 0$ by showing that the right-hand and left-hand limits are both 0.

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \qquad |x| = x \quad if\ x > 0$$

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0 \qquad |x| = -x \quad if\ x < 0$$

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Ex 10: $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{3\sin 3x}{3x} = 3 \lim_{x\to 0} \frac{\sin 3x}{3x} = 3 * 1 = 3$

EX 11:

$$f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

Find

(a) $\lim_{x\to -2} f(x)$ (b) $\lim_{x\to 0} f(x)$ (c) $\lim_{x\to 3} f(x)$

Solution (a): As x approaches -2 from the left, the formula for f is :

$$f(x) = \frac{1}{x+2} \quad \text{so that} \quad \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{x+2} = -\infty$$

As x approaches -2 from the right, the formula for f is

$$f(x) = x^2 - 5$$

so that

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 5) = (-2)^2 - 5 = -1$$

Thus, $\lim_{x \to -2} f(x)$ does not exist.

Solution (b). As x approaches 0 from either the left or the right, the formula for f is

$$f(x) = x^2 - 5$$

Thus,

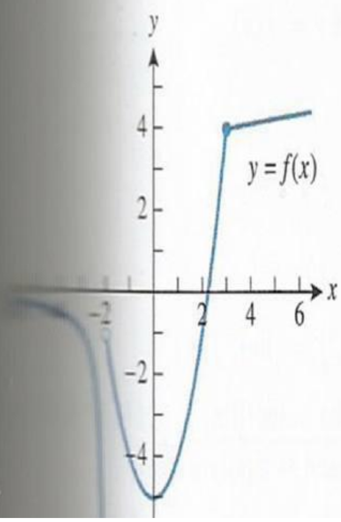
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 5) = 0^2 - 5 = -5$$

Solution (c). As x approaches 3 from the left, the formula for f is

$$f(x) = x^2 - 5$$

so that

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 5) = 3^2 - 5 = 4$$



As x approaches 3 from the right, the formula for f is

$$f(x) = \sqrt{x + 13}$$

so that

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x + 13} = \sqrt{\lim_{x \to 3^+} (x + 13)} = \sqrt{3 + 13} = 4$$

Since the one-sided limits are equal, we have

$$\lim_{x \to 3} f(x) = 4$$